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The Solution of Some Nonlinear Integral Functional Equations

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Abstract:

The integral equation theory is one of mathematical analysis's most important and useful branches. Integral equations occur in a variety of applications, often being obtained from a differential equation, the reason for doing this is that it may make a solution to the problem easier or sometimes, enable us to prove fundamental results on the existence and uniqueness of the solutions. On the other hand, fractional calculus plays an important role in our field of integral equations, and many physical problems can be transformed into integral equations with fractional order. The fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bioengineering and others.

This paper presents the existence theorems of monotonic solutions for nonlinear functional integral equations by using the Darbo fixed point theorem associated with the Hausdorff measure of noncompactness

1. INTRODUCTION

The subject of nonlinear integral equations is considered an important branch of mathematics because it is used for solving many problems such as physics, and chemistry (Cancés and B.Mennucci,1998). In this paper, we will use the technique of measures of non-compactness and Darbo fixed point theory to prove the existence theorem for a nonlinear integral equation in the spaces $L^1(R_+)$. Also, as applications we discuss the existence of solutions for some non-linear integral equations with fractional order, which extends to some previous results in the literature (J.Banaś and W. G. El-Sayed,1980).

2- Notation and auxiliary facts

Let R be the field of real numbers, R_+ be the interval [0,∞) and L^1, be the space of lebsgue integrable functions on a measurable subset [0,∞) of R, with the standard norm.

$$\|x\|_{L^1(R_+)} = \int_0^\infty |x(t)|\, dt$$

One of the most important operator studied in nonlinear functional analysis is the so-called superposition operator (Adam loverro,2004) .

Assume that a function $f(t,x) = f : R_+ \times R \to R$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in R$ and continuous in x for almost all $t \in [0,\infty)$.

Then to every function $x(t)$ being measurable on I we may assign the function

$$(Fx)(t) = f\big(t, x(t)\big), \quad t \in [0,\infty).$$

the operator F in such a way is called the superposition operator generated by the function f.

We have the following theorem due to Appell and Zabrejko (Adam Loverro,2004).

Theorem 2.1:

The superposition operator F generated by the function f maps continuously the space L^1 $into$ L^1if and only if $|f(t,x)| \le a_1(t) + b|x| \;\; \forall\, t \in I$ and $x \in R$ where $a(t) \in L^1$ and $b \ge 0$.

Next, we will mention a desired theorem concerning the compactness in measure of a subset X Of $L^1(R_+)$ (Banaś and W. G. El-Sayed, 1993).

Theorem 2.2:

Let X be an abounded sub-set of $L^1(0,\infty]$ consisting of functions which are almost everywhere nondecreasing (or nonincreasing) on the interval [0,∞). Then X is compact in measure.

Furthermore, we recall a few facts about the convolution operator (Rudin, W.,1966) .

Let $k \in L_1(R)$ be a given function. Then for any function $x \in L_1$, the integral

$$(Kx)(t) = \int_0^\infty k(t - s)\, x(s)\, ds\,,$$

exists for almost every $t \in R_+$. Moreover, the function $(Kx)(t)$ belongs to the space L_1. Thus K is a linear operator which maps the space L_1 into L_1 and K is also bounded since

$\|K\|_{L_1(R)} \le \|K\|_{L_1(R)} L1(R)\|x\|$, for every $x \in L_1$; so, it will be continuous.

Hence the norm $\|K\|$ of the convolution operator is majored by $\|K\|_{L_1(R)}$.

In the sequel, we have the following theorem due to (Krzyz, 1952).

Theorem 2.3:

Assume that $k(t,s) = k\colon R_+^2 \to R$ is measurable on R_+ such that the integral operator,

$$(Kx)(t) = \int_0^\infty k(t,s)x(s)ds\,, \quad t \ge 0\,,$$

maps L^1 into itself the K transforms the set of nonincreasing functions from L^1 into itself if and only if for any $A > 0$, the following implication is true.

$$t_1 < t_2 \to \int_0^A k\,(t_1,s)ds \ge \int_0^A k\,(t_2,s)ds.$$

In the case of space $L^1(0,1)$ we will use the following corollary

Corollary 2.1:

Let $k_i\colon (0,1)^2 \to R_+$ be a measurable function generated by the Fred-Holm operator K acting from $L^1(0,1)$into $L^1 1(0,1)$, if for every $p \in (0,1)$and for all $t_1, t_2 \in (0,1)$ the implication holds,

$$t_1 < t_2 \to \int_0^p k_i\,(t_1,s)ds \ge \int_0^p k_i\,(t_2,s)ds.$$

Finally, we give a short note on measures of noncompactness and fixed point theorem.

Let E be an arbitrary Banach space with $\|.\|$ and the zero element θ.

Let also X be a nonempty and bounded subset of E and B_r be a closed ball in E centered at $\theta$and radius r.

The Hausdorff measure of noncompactness $\chi(X)$ (Banas, J. and Goebel, K.,1980)is defined as

$\chi(X) = inf\,\{r > 0\colon there\ exists\ a\ finite\ subset\ Y\ of\ E\ such\ that\ x \subset Y + B_r\}$.

Another measure was defined in the space L_1 (Banaś and W. G. El-Sayed, 1980), for any $\varepsilon > 0$, let

$$c(X) = lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup\left[\int_D |x(t)|dt, D \subset R_+, meas\,(D) \le \varepsilon\right]\}\}\,,$$

and

$$d(X) = \lim_{T\to\infty}\{\sup\left[\int_T^{\infty}|x(t)|dt, x \in X\right]\}$$

Where meas D denotes the Lebesgue measure of sub set D.

put

$$\gamma(X) = c(X) + d(X).$$

Then we have the following theorem (Banaś and W. G. El-Sayed, 1980),which connects between the two measures

$\chi(X)$ and $\gamma(X)$.

Theorem 2.4:
Let X be a nonempty, bounded and compact in-measure sub set of $L^1(R_+)$, then

$$\chi(X) \leq \gamma(X) \leq 2\chi(X) .$$

In the case of space $L^1(0,1)$ we have the following theorems (Banaś and W. G. El-Sayed, 1980).

Theorem 2.5:
Let X be abounded subset of $L^1(0,1)$ and suppose that there is a family of measurable subset $\{\Omega_c\}_{0\leq c\leq mea\ I}$, of the interval I such that $meas\Omega_c = c\ for\ evrey\ \ c \in [0, meas\ I]$
And for every $x \in X$: $cx(t_1) \leq x(t_2), (t_1 \in \Omega_c, t_2 \notin \Omega_c)$
Then the set X is compact in measure.

Theorem 2.6:
Let X be an arbitrary non-empty and bounded subset of $L^1(0,1)$.If X is compact in measure then $\beta(x) = \chi\ (X)$.
As an application of measures of noncompactness, we recall the fixed point theorem due to(Darbo, G.,1955).

Theorem 2.7:
Let Q be a non-empty, bounded, closed and convex subset of E and let $A: Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of non-compactness, i.e there exist $k \in [0,1)$
such that $\mu(A(X) \leq k\mu(X)$ for any nonempty subset X of, then A has at least one fixed point$Q$.

3-Existence of at least a solution for a nonlinear integral equation on $\boldsymbol{L^1[0,1]}$
Now we will discuss the solvability for the following nonlinear integral equation
$x(t) = g(t) + \int_0^1 k_1(t,s)f_1(s, \int_0^s k_2(s,\tau)f_2(\tau, x(\tau))d\tau)ds,\ t \in [\mathbf{0}, \mathbf{1}]$ (3.1)
in the space $\boldsymbol{L^1}[\mathbf{0}, \mathbf{1}]$.
We shall treat equation (3.1) under the following assumptions which are listed below:
(i) $g \in \boldsymbol{L^1}[\mathbf{0}, \mathbf{1}]$ and almost everywhere positive and nonincreasing in $\boldsymbol{L^1}[\mathbf{0}, \mathbf{1}]$.
(ii) $f_i: [0,1] \times R \to R \quad, i = 1,2$ are nonincreasing functions on $R_+$with respect to t and x , satisfy carathéodory conditions, there are two functions $a_i \in L^1(R_+)$ and two Constants $b_i \geq 0$, such that:
$|f_i(t,x)\ | \leq a_i(t) + b_i|x|$, for all $t \in R_+, x \in R$ and $f_i(t,x) \geq 0, \forall x \geq 0,\ i = 1,2$
(iii) $k_i[: [0,1]\ \times, \to R, i = 1,2$, are measurable with respect to t and s and $K_i: L^1 \to L^1$ is bounded with norm $\|K\|$,
Also, $\forall A > 0$ and for all $t_1, t_2 \in [0,1]$, we have :
$$t_1 < t_2 \to \int_0^A k_i(t_1,s)ds \geq \int_0^A k_i(t_2,s)ds \qquad i = 1,2$$
(iv) $b_1b_2\|K_1\|\|K_2\| < 1$.
Then we can prove the following theorem

Theorem 3.1:
Let the assumptions (i) $-$(iv) be satisfied, then the equation (3.1) has at least one solution ,$x \in L^1[0,1]$ being almost everywhere non increasing on$[0,1]$.

Proof

Consider the operator H:

$Hx(t) = g(t) + \int_0^1 k_1(t,s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) ds$

The equation (3.1) takes the form

$$x(t) = Hx(t)$$

First, let $x \in L^1[0,1]$

Then using our assumption (i)→(iii), we have

$|Hx(t)| \leq |g(t)| + \left| \int_0^1 k_1(t,s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| ds$

$\int_0^1 |Hx(t)|\, dt \leq ||g|| + ||K_1F_1K_2F_2x||$

$$\begin{aligned}
\leq ||g|| + ||K_1|| ||K_1F_1K_2F_2x|| \\
\leq ||g|| + ||K_1|| \int_0^1 |f_1(s, \int_0^t k_2(s,t) f_2(t, x(t)) ds)|\, dt \\
\leq ||g|| + ||K_1|| \int_0^1 [a_1(s) + b_1 | \int_0^t k_2(s,t) f_2(t, x(t)) ds|]\, dt \\
\leq ||g|| + ||K_1|| [\|a_1\| + b_1 \int_0^1 \int_0^t |k_2(s,t) f_2(t, x(t))| ds\,]dt \\
\leq ||g|| + ||K_1|| \left[\|a_1\| + b_1 ||K_2|| \int_0^1 [a_2(t) + b_2 |x(t)|] \right] dt \\
\leq ||g|| + ||K_1|| \left[\|a_1\| + b_1 ||K_2|| [||a_2|| + b_2 ||x||] \right] dt \to (1)
\end{aligned}$$

From the last estimate, the space L^1 into itself using theorem (2.1)

Moreover, using the estimate (1), we see that the operator H transforms the ball B_r into itself, where:

$$r = \frac{||g|| + ||K_1|| \|a_1\| + b_1 ||K_1|| ||K_2|| ||a_2||}{1 - b_1 b_2 ||K_1|| ||K_2||}$$

Let Q_r be subset of B_r consisting of all functions being almost everywhere positive and non-increasing on [0,1].

Note that Q_r is a non-empty, bounded, closed, convex subset of $L^1[0,1]$.

Moreover, given theorem (2.2) the set Q_r is compact in measure.

Next, by taking $x \in Q_r$,

Then $x(t)$ is almost everywhere positive and non-decreasing on R_+, and consequently $K_i x(t)$ is also of the same type (in virtue of the assumption (iii) and theorem (2.3)

Further, the assumption (ii) permits us to deduce that,

$$Hx(t) = g(t) + K_1F_1K_2F_2x(t),$$

Is almost everywhere positive and non-decreasing on [0,1], this fact together with assertion $H: B_r \to B_r$, gives that self-mapping of the set Q_r, since the

Operator K is continuous and F is continuous in view theorem (2.1), we conclude that H maps continuously Q_r into Q_r.

Finally, assume that X is non-empty subset of Q_r and $\epsilon > 0$ is fixed, then for an arbitrary $x \in X$ and for a set $D \subset [0,1]$, meas $D \leq \epsilon$, we obtain

$$\begin{aligned}
\int_D |(Hx)(t)|\, dt = \int_D \left| g(t) + \int_0^1 k_1(t,s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) \right| ds dt \\
\leq \int_D |g(t)|\, dt + \int_D \left| \int_0^1 k_1(t,s) f_1(s, \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau) ds \right| dt \\
\leq \int_D |g(t)|\, dt + \int_D \int_0^1 |k_1(t,s)| \left| \left[a_1(s) + b_1 \right| \int_0^s k_2(s,\tau) f_2(\tau, x(\tau)) d\tau \right|] ds dt \\
\leq ||g||_{L^1(D)} + \int_D \int_0^1 |k_1(t,s)| \left[a_1(s) + b_1 \int_0^s |k_2(s,\tau)| [a_2(\tau) + b_2 |x(\tau)|] d\tau ds \right] dt \\
\leq ||g||_{L^1(D)} + \|K_1\|_D ||a_1||_{L^1(D)} + b_1 \|K_1\|_D \|K_2\|_D ||a_2||_{L^1(D)} + b_1 b_2 \|K_1\|_D \|K_2\|_D \int_D |x(s)|\, ds
\end{aligned}$$

Where, $K: L^1(D) \to L^1(D)$, as simple consequently, we get

$\int_D |(Hx)(t)|\, dt \leq ||g||_{L^1(D)} + \|K_1\|_D ||a_1||_{L^1(D)} + b_1 \|K_1\|_D \|K_2\|_D ||a_2||_{L^1(D)} + b_1 b_2 \|K_1\|_D \|K_2\|_D \int_D |x(s)|\, ds$

The last in quantity gives,

$\lim_{\epsilon \to 0} \{ \sup [\int_D |g(t)| dt + \|K_1\| \int_D |a_1(t)| dt + b_1 ||K_1||_D ||K_2||_D \int_D |a_2(t)| dt : D \subset [0,1], \text{meas } D \leq \epsilon]\} = 0$

Then $\beta(HX) \leq b_1 b_2 ||K_1|| ||K_2|| \beta(X)$ (2)

Further, more fixing $T > 0$ we arrive at the following estimate
$\int_T^\infty |Hx(t)|dt \leq \int_T^\infty |g(t)|dt + \|K_1\| \int_T^\infty |a_1(t)|dt + b_1\|K_1\|_D\|K_2\|_D \int_T^\infty |a_2(t)|\, dt \ + b_2b_2\|K_1\|_D\|K_2\|_D \int_T^\infty |x(t)|dt$
Since $\lim_{T\to\infty} T = \infty$, the above in quality gives
$d(HX) \leq b_1b_2\left||K_1|\right|_D\left||K_2|\right|_D$ (3)
Hence, by combining (2) and (3) we get
$\gamma(HX) \leq b_1b_2\left||K_1|\right|\left||K_2|\right|\gamma(X)$
Where γ denotes the measure of non-compactness since Q_r is compact in measure, then by using theorem (2.4).
The last inequality together with the assumption (iv), enables us to apply theorem (2.7), Which proves the existence of a fixed point for the operator H in Q_r. ■

4. Results

4-existence of at least a solution for the nonlinear integral equations with fractional order:
In this section, we will discuss solvability for the following nonlinear integral equation with fractional order in $L^1[0,1]$

$$x(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)\, ds, \quad t \in [0,1] \qquad (4.1)$$

We shall treat equation (4.1) under the following assumptions which are listed below:

(i) $g \in L^1$ [0,1],and almost everywhere positive and non-decreasing in $L^1[0,1]$.
(ii) f_i $[0,1]:\times R \to R, i = 1,2$, are non-decreasing functions on [0,1] concerning t and x
Satisfy Carathéodory conditions, there are two functions $a_i \in L^1$ [0,1] and two constants $b_i \geq 0$ such that:
$|f_i(t,x)| \leq a_i(t) + b_i|x|$, for all $t \in [0,1], x \in R$ and $f_i(t,x) \geq 0, \forall x \geq 0, i = 1,2$
(iii) $k: [0,1] \times [0,1] \to R$, is measurable concerning t and s and $K: L^1 \to L^1$ is bounded by the norm $\left||K|\right|$.
Also, $\forall A > 0$ and for all $t_1, t_2 \in R_+$, we have

$$t_1 < t_2 \to \int_0^A (t_1 - s)^{\alpha-1}ds \geq \int_0^A (t_2 - s)^{\beta-1}ds$$

(iv) $\frac{b_1b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1$
Then we can prove the following theorem,

Theorem 4.1:
Let the assumptions (i)-(iv) be satisfied, then the equation (4.1) has at least one solution, $x \in L^1[0,1]$being almost everywhere non-decreasing on [0,1].
Proof
Consider the operator H:

$$Hx(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)\, ds, \qquad t \in [0,1]$$

Where $0< \alpha \leq 1$ and $0 < \beta \leq 1$
Then the equation (4.1) takes the form

$$x(t) = Hx(t)$$

First, let $x \in L^1[0,1]$
Then using our assumption (i)-(iii) we have,

$$|Hx(t)| \leq |g(t)| + \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)\right| ds$$

$$\int_0^1 |Hx(t)|dt \leq \int_0^1 |g(t)|dt + \int_0^1 \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)\right|dsdt$$

$$\leq \left||g(t)|\right| + \int_{s=0}^1 \int_{t=s}^{t=1} \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| |f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)|dtds$$

$$\leq \left||g|\right| + \int_{s=o}^1 \frac{1}{\Gamma(\alpha+1)} |f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)d\tau)|ds$$

$$\ll \|g\| + \frac{1}{\Gamma(\alpha+1)} \|F_1KF_2\|$$

$$\leq \left||g|\right| + \frac{1}{\Gamma(\alpha+1)} \int_0^1 [a_1(s) + b_1| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau))|d\tau)]ds$$

$$\leq \left||g|\right| + \frac{1}{\Gamma(\alpha+1)} [\left||a_1|\right| + b_1 \int_0^1 \int_0^s \left|\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2\big(\tau, x(\tau)\big)\right|d\tau]ds$$

$$\leq \left||g|\right| + \frac{1}{\Gamma(\alpha+1)} [\left||a_1|\right| + b_1 \frac{1}{\Gamma(\beta+1)} \int_0^1 [a_2(\tau) + b_2|x(\tau)|]d\tau]$$

$$\leq \left||g|\right| + \frac{1}{\Gamma(\alpha+1)} [\left||a_1|\right| + \frac{b_1}{\Gamma(\beta+1)} [\|a_2\| + b_2\left||x|\right|] \to (1)$$

From the last estimate, we deduce that the operator H maps continuously, the space L^1 into itself using theorem (2.1). Moreover, using the estimate (1). we see that the operator H transforms the ball B_r into itself where :

$$r = \frac{||g|| + \frac{1}{\Gamma(\alpha+1)}[\|a_1\| + \frac{b_1}{\Gamma(\beta+1)}||a_2||]}{\left(1 - \frac{b_1 b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)}$$

Let $Q_r$be subset of B_r consisting of all functions being almost everywhere positive and non-increasing on R_+.
Note that Q_r is a non-empty, bounded , closed, convex subset of $L^1[0,1]$.
Moreover, in view of Theorem (2.2) the set Q_r is compact in measure.
Next, by taking $x \in Q_r$, then $x(t)$ is almost everywhere positive and non-increasing on $L^1[0,1]$.and consequently $Kx(t)$ is also of the same type (in virtue of the assumption (iii) and theorem (2.3)).
Further, the assumption (ii) permits us to deduce that:

$$Hx(t) = g(t) + F_1 K F_2 x(t)$$

Is also almost everywhere positive and non-decreasing on R_+, this fact together with assertion, $H: B_r \to B_r$ gives that self–mapping of the set Q_r.
Since the operator K is continuous and F is continuous in view theorem (2.1), we conclude that H mapps continuous Q_r into Q_r .
Note, that:

$$K_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$$

$$K_1 x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds$$

$$||K_1 x|| = \int_{t=0}^1 \int_{s=0}^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| |x(s)| ds dt$$

$$= \int_{s=0}^1 \int_{t=s}^1 \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| |x(s)| dt ds$$

Let $J = \int_{t=s}^1 \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| dt$

$$= \int_{t-s=0}^1 \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right| d(t-s) = 1$$

Then

$$||K_1 x|| = \frac{1}{\Gamma(\alpha+1)} \int_0^1 |x(s)|\, ds$$

$$||K_1|| = \frac{\|x\|}{\Gamma(\alpha+1)}$$

$$\sup_{\substack{x \neq 0 \\ x \in k}} \frac{\|Kx\|}{\|x\|} = 1$$

Also, we can note that

$$K_2(s,\tau) = \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}$$

$$K_2 x(t) = \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} x(\tau) d\tau$$

$||K_2 x|| = \int_{\tau=0}^1 \int_{s=\tau}^t \left|\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\right| |x(\tau)| d\tau ds$Then

$$||K_2 x|| = \frac{1}{\Gamma(\beta+1)} \int_{\tau=0}^1 |x(\tau)|\, d\tau$$

$$||K_2 x|| = \frac{\|x\|}{\Gamma(\beta+1)}$$

Then

$||K_2|| = \frac{1}{\Gamma(\beta+1)}$

Finally, assume that X is no-empty subset of Q_r and $\epsilon > 0$ is fixed, then for an arbitrary $x \in X$ and for a set $D \subset [0,1]$, meas $D \le \epsilon$, we obtain

$$\int_D |(Hx)(t)|\, dt = \int_D \left| g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau)) d\tau) \right| dsdt$$

$$\le \int_D |g(t)|\, dt + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau)) d\tau) ds \right| dt$$

$$\le \int_D |g(t)|\, dt + \int_D \int_{t=s}^1 \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right| [a_1(s) + b_1 \left| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} f_2(\tau, x(\tau)) d\tau \right|] dtds$$

$$\le ||g||_D + \frac{1}{\Gamma(\alpha+1)} [||a_1|| + b_1 \frac{1}{\Gamma(\beta+1)} \int_D [a_2(\tau) + b_2 |x(\tau)|] d\tau]$$

$$\le ||g||_D + \frac{1}{\Gamma(\alpha+1)} ||a_1|| + \frac{b_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} ||a_2|| + \frac{b_1 b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_D |x(\tau)| d\,\tau$$

Where $K: L^1(D) \to L^1(D)$, as simple consequently, we get

$$\int_D |(Hx)(t)|\, dt \le ||g||_{L^1(D)} + \frac{1}{\Gamma(\alpha+1)} ||a_1||_{L^1(D)} + \frac{b_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} ||a_2||_{L^1(D)} + \frac{b_1 b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_D |x(\tau)| d\tau$$

The above in quantity gives

$$\lim_{\epsilon \to 0} \{ \sup [\int_D |g(t)| dt + \frac{1}{\Gamma(\alpha+1)} \int_D |a_1(t)| dt + \frac{b_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_D |a_2(t)| dt:\ D \subset [0,1],\ \text{meas}\ D \le \epsilon] \} = 0$$

Then $\beta(Hx(t)) \le \frac{b_1 b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \beta(X) \to (2)$

Where β denotes the measure of non-compactness since Q_r is compact in measure, then by using theorem (2.4).
We can write the last inequality in the form

$$\chi(Hx(t)) \le \frac{b_1 b_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \chi(X)$$

The last inequality together with the assumption (iv), enables us to apply theorem (2.7), Which proves the existence of a fixed point for the operator H in Q_r. ∎

In the same way, we will discuss solvability for the following linear integral equation with fractional order on the space $L^1(0,1)$.

$$x(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)\, ds, \quad t \in [0,1] \qquad (4.2)$$

We shall treat equation (4.2) under the following assumptions which are listed below:

(i) $g \in L^1(0,1)$, and almost everywhere positive and non-decreasing in (0,1),

(ii) $f_i : (0,1) \times R \to R, i = 1,2$, are non-decreasing functions on (0,1) concerning t and x satisfy Carathéodory conditions, there are two functions $a_i \in L^1(0,1)$ and two constants $b_i \ge 0$ such that:
$|f_i(t,x)| \le a_i(t) + b_i|x|$, for all $t \in (0,1), x \in R$ and $f_i(t,x) \ge 0, \forall x \ge 0, i = 1,2$

(iii) $k: (0,1) \times (0,1) \to R_+$, is measurable concerning t and s and $K: L^1 \to L^1$.
(From assumption (iii), we see that K is continuous and so it is bounded by norm $||K||$).
Also, $\forall A > 0$ and for all $t_1, t_2 \in (0,1)$, we have

$$t_1 < t_2 \to \int_0^A (t_1 - s)^{\alpha-1} ds \ge \int_0^A (t_2 - s)^{\alpha-1} ds$$

(iv) $\frac{b_1 b_2 ||K||}{\Gamma(\alpha+1)} < 1$

Then we can prove the following theorem,

Theorem 4.1:
Let the assumptions (i)-(iv) are satisfied, then the equation (4.1) has at least one solution, $x \in L^1(0,1)$ being almost everywhere non-decreasing on (0,1).

Proof
Consider the operator H:
$Hx(t) = g(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)\, ds, \qquad t \in [0,1]$
Where $o < \alpha \leq 1$

Then the equation (4.2) takes the form
$$x(t) = Hx(t)$$
First, let $x \in L^1[0,1]$
Then using our assumption (i)-(iii) we have,
$|Hx(t)| \leq |g(t)| + \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)\right| ds$
$\int_0^1 |Hx(t)| dt \leq \int_0^1 |g(t)| dt + \int_0^1 \int_0^t | \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds dt$
$\leq ||g(t)|| + \int_{s=0}^1 \int_{t=s}^1 | \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} || f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| dt ds$
$\leq ||g|| + \int_{s=o}^1 | \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(s-s)^{\alpha}}{\Gamma(\alpha+1)} || f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$
$\leq ||g|| + \int_{s=o}^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} | f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$
$\leq ||g|| + \int_{s=o}^1 \frac{1}{\Gamma(\alpha+1)} | f_1(s, \int_0^s k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$
$\leq ||g|| + \frac{1}{\Gamma(\alpha+1)} \int_0^1 | f_1(s, \int_0^\tau k(s,\tau) f_2(\tau, x(\tau)) d\tau)| ds$
$\leq ||g|| + \frac{1}{\Gamma(\alpha+1)} ||F_1 K F_2||$
$\leq ||g|| + \frac{1}{\Gamma(\alpha+1)} \int_0^1 [a_1(s) + b_1| \int_0^s k(s,\tau) f_2(\tau, x(\tau))| d\tau)] ds$
$\leq ||g|| + \frac{1}{\Gamma(\alpha+1)} [||a_1|| + b_1 \int_0^1 \int_0^s |k(s,\tau) f_2(\tau, x(\tau))| d\tau] ds$
$\leq ||g|| + \frac{1}{\Gamma(\alpha+1)} [||a_1|| + b_1 ||K|| [||a_2|| + b_2 ||x||]] \rightarrow (1)$
From the last estimate we deduce that the operator H maps continuously, the space L^1 into itself using theorem (2.1).
Moreover, using the estimate (1), we see that the operator H transforms the ball B_r into itself where:

$$r = \frac{||g|| + \frac{1}{\Gamma(\alpha+1)} [||a_1|| + b_1 ||K|| ||a_2||]}{(1 - \frac{b_1 b_2 ||K||}{\Gamma(\alpha+1)})}$$

Let Q_r be subset of B_r consisting of all functions being are almost everywhere positive and non-increasing on $(0,1)$.
Note that Q_r is non-empty , bounded , closed, convex subset of $L^1(0,1)$.
Moreover, in view of Theorem (2.5) the set Q_r is compact in measure.
Next, by taking $x \in Q_r$, then $x(t)$ is almost everywhere positive and non-increasing on (0, 1) and consequently $Kx(t)$ is also of the same type (in virtue of the assumption (iii) and theorem (2.1)).
Further, the assumption (ii) permits us to deduce that:

$$Hx(t) = g(t) + \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} F_1 K F_2 x(t)$$
Is also almost everywhere positive and non-decreasing on (0, 1), this fact together with assertion, $H: B_r \rightarrow B_r$ gives that self–mapping of the set Q_r.
since the operator K is continuous and F is continuous in view theorem (2.1), we conclude that H mapps continuous Q_r into Q_r .

Note, that:

$K(t,s)=\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$

$Kx(t)=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds$

$||Kx||=\int_{t=0}^1\int_{s=0}^t \frac{|t-s|^{\alpha-1}}{\Gamma(\alpha)}|x(s)|dsdt$

$\quad =\frac{1}{\Gamma(\alpha+1)}\int_{s=0}^1|x(s)|ds$

$||Kx||=\frac{||x||}{\Gamma(\alpha+1)}$

$\sup_{\substack{x\neq 0\\ x\in k}}\frac{\|Kx\|}{\|x\|}=\frac{1}{\Gamma(\alpha+1)}$

$||K||=\frac{1}{\Gamma(\alpha+1)}$

Finally, assume that X is no-empty subset of Q_r and $\epsilon>0$ is fixed, then for an arbitrary $x\in X$ and for a set $D\subset(0,1)$, meas $D\le\epsilon$, we obtain

$\int_D |(Hx)(t)|\,dt=\int_D \left|g(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,\int_0^s k(s,\tau)f_2\big(\tau,x(\tau)\big)d\tau)\right|dsdt$

$\le\int_D |g(t)|\,dt+\int_D \left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,\int_0^s k(s,\tau)f_2\big(\tau,x(\tau)\big)d\tau)ds\right|dt$

$\le\int_D |g(t)|\,dt+\int_D \int_{t=s}^1\left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right|[a_1(s)+b_1|\int_0^s k(s,\tau)f_2\big(\tau,x(\tau)\big)d\tau|]dtds$

$\le||g||_D+\frac{1}{\Gamma(\alpha+1)}[||a_1||+b_1||K||\int_D [a_2(\tau)+b_2|x(\tau)|]ds]$

$\le||g||_D+\frac{1}{\Gamma(\alpha+1)}||a_1||+\frac{1}{\Gamma(\alpha+1)}b_1||K||||a_2||+\frac{b_1b_2||K||}{\Gamma(\alpha+1)}\int_D |x(s)|ds$

Where $K: L^1(D)\to L^1(D)$, as simple consequently we get

$\int_D |(Hx)(t)|\,dt\le||g||_{L^1(D)}+\frac{1}{\Gamma(\alpha+1)}||a_1||_{L^1(D)}+\frac{1}{\Gamma(\alpha+1)}b_1||K||_D||a_2||_{L^1(D)}+\quad\frac{1}{\Gamma(\alpha+1)}b_1b_2\|K\|\int_D |x(s)|ds$

The above in quantity gives

$\lim_{\epsilon\to 0}\{\sup[\int_D |g(t)|dt+\frac{1}{\Gamma(\alpha+1)}\int_D |a_1(t)|dt++\frac{1}{\Gamma(\alpha+1)}b_1||K||_D\int_D |a_2(t)|dt:$
$\quad D\subset(0,1),\ \text{meas}\ D\le\epsilon]\}=0$

Then $\beta(Hx(t))\le\frac{b_1b_2||K||}{\Gamma(\alpha+1)}\beta(X)$

Where β is the De Blasi measure of non-compactness: Since Q_r is compact in measure, then by using theorem (2.6), we can write the last inequality in the form

$$\chi(HX)\le\frac{b_1b_2||K||}{\Gamma(\alpha+1)}\chi(X)$$

This inequality together with the assumption (vi) enables us to apply theorem (2.8), which proves the existence of a fixed point for the operator H in Q_r. ■

5. CONCLUSION

In this work, we determined the sufficient conditions under which the existence theorem of a nonlinear integral equation with convolution kernel is proved in the space $L^1[0,1]$, Also the same situation is proved for a nonlinear integral equation with fractional order in the spaces $L^1[0,1]$ and $L^1(0,1)$

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